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OPTIMAL PARAMETRIC STABILIZATION OF AN INVERTED PENDULUM*

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Problem of optimal stabilization of the unstable upper position of equilibrium of a pendulum is studied using the external periodic forces. In the first case the force is applied to the hinge from which the pendulum is suspended (the vibrating suspension point /1—3/), and in the second case the force moments are applied to the rods clamped at the uppermost point of the pendulum. The problem of determining such forces (or force moments) is solved for a given class of functions ensuring the optimal (in the sense of the minimum of the general index /4/) stabilisation of the pendulum and restricting, in addition, within the given limits, the displacement of the vibrating elements of the construction.

1. Equations of motion. Pendulum with a moving point of suspension. Figure 1 shows two masses M and m in the XOY plane, connected by a weightless rigid rod of length l. The mass m can move without friction along the y-axis in a groove containing a gap which enables a free motion of the rod. A hinge at the point m restricts the motion of the rod to the XOY plane. The generalized coordinates of the system in question are φ and y where φ is the angle between the rod and positive direction of the y-axis, and y is the ordinate of the moving mass m. The external forces acting on the system are: gravitational forces $(0, -Mg)$ and $(0, -mg)$ applied, respectively, to the masses M and m, the controlling force $(0, F(t))$ to the mass m, and the rotational force of friction in the hinge m generating a moment relative to the hinge axis and proportional to $\varphi^{\cdot}$.

We write the kinetic energy of the system and the generalized forces ($k>0$ is the coefficient of friction), as follows:

$$L_1 = {}^1/_2 m y^{\cdot 2} + {}^1/_2 M\,[l^2\cos^2\varphi\varphi^{\cdot 2} + (y^{\cdot} - l\sin\varphi\varphi^{\cdot})^2],\quad Q_y = -(m+M)g + F(t),\quad Q_\varphi = Mgl\sin\varphi - k\varphi^{\cdot}$$

and the Lagrange method yields the following equations of motion:

$$(m+M)y^{\cdot\cdot} - Ml\,(\cos\varphi\varphi^{\cdot 2} + \sin\varphi\varphi^{\cdot\cdot}) = -(m+M)g + F(t) \tag{1.1}$$

$$Ml^2\varphi^{\cdot\cdot} - Mly^{\cdot\cdot}\sin\varphi = Mgl\sin\varphi - k\varphi^{\cdot} \tag{1.2}$$

Pendulum with moving rods. Figs.2 and 3 depict, in the system of fixed $OXYZ$ axes, a rod hinged at O. The axis of this hinge is directed along the z-axis, and ensures that the rod OD moves within the XOY plane. The rods DA and DA' are hinged at the point D. The axis of the hinges supporting the rods DA and DA' lies in the XOY plane and is either parallel to the x-axis (Fig.2), or perpendicular to the rod OD (Fig.3). The rods DA and DA' are assumed homogeneous, rigid and identical, each of length $2l$ and mass m. The homogeneous, rigid rod OD has mass M and length L.

Thus the hinged supports in the system are such, that the rod OD can move only in the XOY plane and the rods DA and DA' either in the plane parallel to YOZ (Fig.2), or in the plane perpendicular to XOY and passing through OD (Fig.3).

In accordance with the construction described we introduce the generalized coordinates of the system: φ is the angle between OD and the positve direction of the y-axis, ψ_1 is the angle between DA and either the vertical, or the line OD (Figs.2 and 3) and ψ_2 is the same angle for the rod DA'. We assume that the following external forces act on the system: the forces of gravity $(0, -Mg, 0)$, $(0, -mg, 0)$, the control moments $\mathbf{M}^{01}(t)$ and $\mathbf{M}^{02}(t)$ applied, respectively, to the rods DA and DA' and directed along the axis of the hinges towards D, and the friction at the hinge O responsible for the moment about the axis of the hinge and proportional to $\varphi^{\cdot}$. We assume, for simplicity, that the motion of the rods DA and DA' is symmetrical with respect to the XOY plane. This a priori holds when $\mathbf{M}^{01}(t) = -\mathbf{M}^{02}(t) = \mathbf{M}^{0}(t)$ and the initial conditions (coordinates and velocities) of DA and DA' are symmetrical with respect to the XOY plane. Such an assumption enables us to replace two generalized coordinates ψ_1 and ψ_2 by a single coordinate $\psi = \psi_1 = -\psi_2$.

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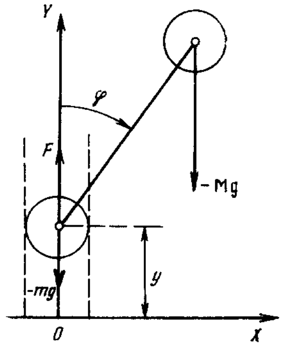

Fig.1

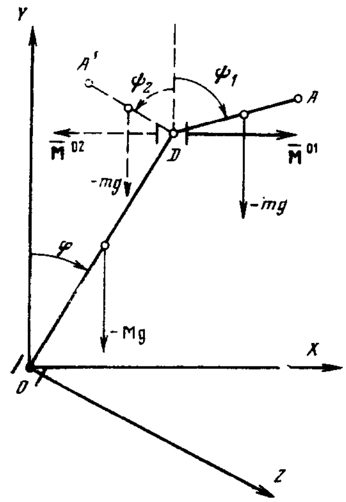

Fig.2

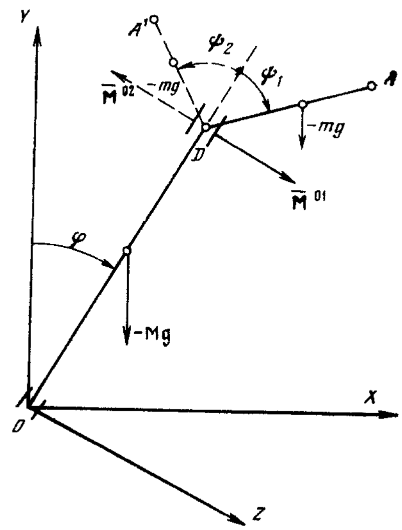

Fig.3

The kinetic energies and generalized forces for the systems shown in Figs.2 and 3 are given, respectively, by

$$L_2 = {}^1\!/_6 ML^2\varphi^{\cdot 2} + 2\,[{}^1\!/_2 mL^2\varphi^{\cdot 2} + mlL \sin\varphi \sin\psi\,\varphi^{\cdot}\psi^{\cdot} + {}^2\!/_3 ml^2\psi^{\cdot 2}]$$
$$Q_\varphi = (MgL/2 + 2mgL)\sin\varphi - k\varphi^{\cdot},\quad Q_\psi = 2[M^\circ(t) + mgl\sin\psi]$$
$$L_3 = {}^1\!/_6 ML^2\varphi^{\cdot 2} + 2\,[{}^2\!/_3\, ml^2\psi^{\cdot 2}\,(1 + \cos^2\psi) + {}^1\!/_2 mL^2\varphi^{\cdot 2} + mlL\cos\psi\,\varphi^{\cdot 2}]$$
$$Q_\varphi = (MgL/2 + 2mgL + 2mgl\,(\cos\psi)\sin\varphi - k\varphi^{\cdot}$$
$$Q_\psi = 2\,[M^\circ(t) + mgl\cos\varphi\sin\psi]$$

The factor 2 preceding the square brackets in the expressions for L_2 and L_3 reflects the presence of two, symmetrically moving rods DA and DA'.

The Lagrange equations for the construction shown in Fig.2 are

$$(ML/3 + 2mL)\varphi^{\cdot\cdot} + k\varphi^{\cdot}/L + 2ml\,(\sin\varphi\cos\psi\,\psi^{\cdot 2} + \sin\varphi\sin\psi\,\psi^{\cdot\cdot}) = (M/2 + 2m)g\sin\varphi \qquad (1.3)$$

$$ {}^4\!/_3 ml^2\psi^{\cdot\cdot} + mlL\,(\cos\varphi\sin\psi\,\varphi^{\cdot 2} + \sin\varphi\sin\psi\,\varphi^{\cdot\cdot}) = M^\circ(t) + mgl\sin\psi \qquad (1.4)$$

and for the construction in Fig.3,

$$(ML/3 + 2mL + 4ml\cos\psi)\varphi^{\cdot\cdot} + (k/L - 4ml\sin\psi\,\psi^{\cdot})\varphi^{\cdot} = (M/2 + 2m\,(1 + l\cos\psi/L))g\sin\varphi \qquad (1.5)$$

$$ {}^4\!/_3 ml^2\,(1 + \cos^2\psi)\psi^{\cdot\cdot} - {}^4\!/_3\, ml^2\psi^{\cdot 2}\cos\psi\sin\psi + 2mlL\sin\psi\,\varphi^{\cdot 2} = M^\circ(t) + mgl\cos\varphi\sin\psi \qquad (1.6)$$

2. Formulation of the problem. Pendulum with movable point of suspension.

We shall consider the motions described by the equations (1.1) and (1.2) in the neighborhood of the point $\varphi = \varphi^{\cdot} = 0$. Neglecting in (1.1) the terms of first and higher order of smallness in φ and $\varphi^{\cdot}$, we obtain

$$y^{\cdot\cdot} = -g + \frac{F(t)}{m + M} \qquad (2.1)$$

Let us neglect in (1.2) the terms of second and higher order of smallness in φ and substitute $y^{\cdot\cdot}$ from (2.1). This yields

$$\varphi^{\cdot\cdot} + k_1\varphi^{\cdot} = \frac{F(t)}{l\,(m + M)}\,\varphi,\quad k_1 = \frac{k}{Ml^2} \qquad (2.2)$$

Let the control force $F(t)$ satisfy the restrictions

$$-F_1 \leqslant F(t) \leqslant F_2,\quad F_1 > 0,\quad F_2 > 0,\quad t \in [0, \infty) \qquad (2.3)$$

We consider the period $T>0$, and pose the following problem: to find a T-periodic function $F(t)$ satisfying the conditions (2.3), ensuring the best (in the sense of a minimum value of the general index) stability of the solutions of (2.2) and such, that the equation (2.1) has a T-periodic solution lying in the prescribed neighborhood $|y|\leqslant\varepsilon_0$, $\varepsilon_0>0$.

We note that in /1—3/ the motion of the point of suspension was defined **a priori** by $y(t)=\varepsilon\sin\omega t$ or some other periodic function. Therefore the problem of determining the optimal control forces did not arise.

Pendulum with moving rods. We shall consider the motion of the constructions depicted in Figs.2 and 3 near the point $\varphi=\varphi^{\cdot}=0$, $\psi=\pi/2$. Thus, we shall investigate the motion of the rod OD near the vertical, with the rods DA and DA' oscillating near the horizontal. Linearizing the equations (1.3)—(1.6) as before as carrying out the elementary transformations, we obtain

$$\left(\frac{ML}{3}+2mL\right)\varphi^{\cdot\cdot}+\frac{k}{L}\varphi^{\cdot}=\left[\frac{M+m}{2}g-\frac{3}{2}\frac{M^{\circ}(t)}{l}\right]\varphi \tag{2.4}$$

$$\frac{4}{3}ml^2\Delta\psi^{\cdot\cdot}=-M^{\circ}(t)-mgl,\quad \Delta\psi\equiv\frac{\pi}{2}-\psi \tag{2.5}$$

$$\left(\frac{ML}{3}+2mL\right)\varphi^{\cdot\cdot}+\left(\frac{k}{L}+4ml\Delta\psi^{\cdot}\right)\varphi^{\cdot}=\left(\frac{M}{2}+2m\right)g\varphi \tag{2.6}$$

Linearization of (1.6) yields (2.5).

Let the control moment $M^{\circ}(t)$ satisfy the restrictions

$$-M_1^{\circ}\leqslant M^{\circ}(t)\leqslant M_2^{\circ},\ M_1^{\circ}>0,\ M_2^{\circ}>0,\ t\in[0,\infty) \tag{2.7}$$

We consider the period $T>0$ and pose the following problem: to find T-periodic function $M^{\circ}(t)$ satisfying the conditions (2.7), ensuring the best (in the sens of the minimum value of the general index) stability of the solutions of (2.4) and (2.6) and such, that the equation (2.5) has a T-periodic solution lying in the given neighborhood $|\Delta\psi|\leqslant\varepsilon_0$, $\varepsilon_0>0$.

If OD, DA and DA' are regarded as the body and arms, respectively of man, then the body is stabilized by the periodic up and down motions of the arms. The authors of /5/ describe a case when a man ensures the stabilization of his body by rotating his arms with increasing angular velocity.

3. Formulation and solution of the generalized problem. Let C be the class of all piecewise continuous real scalar functions on $[0,\infty)$. We consider the following subsets of C:

$$R(T,m_1,m_2)=\{u\in C: u(t)\equiv u(t+T),\ -m_1\leqslant u(t)\leqslant m_2,\ t\in[0,\infty)\}$$

$$R_h(T,m_1,m_2)=\left\{u\in R(T,m_1,m_2): \frac{1}{T}\int_t^{t+T}u(\tau)\,d\tau=h\right.$$

$$\left.h\in[-m_1,m_2]\right\}$$

We require to find the function $u\in R(T,m_1,m_2)$, ensuring the minimum value of the general index of the equation

$$x^{\cdot\cdot}=[a+bu(t)]x,\ a>0,\ b>0 \tag{3.1}$$

and such, that the equation

$$y^{\cdot\cdot}=u(t)-c,\ c>0 \tag{3.2}$$

has a T-periodic solution satisfying the condition

$$|y(t)|\leqslant\varepsilon_0,\quad t\in[0,\infty) \tag{3.3}$$

We note that in (3.1)—(3.3) a, b, c, ε_0 are given positive constants and $c\leqslant m_2$ (when $c>m_2$, the equation (3.2) has, as we know, no periodic solutions if $u\in R(T,m_1,m_2)$).

We now proceed to solve the problem in question. We define arbitrarily $h\in[-m_1,m_2]$ and introduce the notation

$$M_1 = a - bm_1, \quad M_2 = a + bm_2$$
$$\sigma_1 = M_2^{1/2}\frac{h - m_1}{m_2 + m_1}, \quad \sigma_2 = |M_1|^{1/2}\frac{m_2 - h}{m_2 + m_1}$$
$$\gamma = \frac{1}{2}\left(\left|\frac{M_2}{M_1}\right|^{1/2} + \left|\frac{M_1}{M_2}\right|^{1/2}\operatorname{sign} M_1\right)$$

The following lemma which was proved in /6/, holds.

Lemma. If $M_1 \geqslant 0$, then the smallest general index of equation (3.1) which can be attained on the functions of $R_h(T, m_1, m_2)$, is

$$\mu^0 = \frac{1}{T}\ln\left(A + \sqrt{A^2 - 1}\right) \tag{3.4}$$
$$A = \operatorname{ch}(\sigma_1 T)\operatorname{ch}(\sigma_2 T) + \gamma\operatorname{sh}(\sigma_1 T)\operatorname{sh}(\sigma_2 T)$$

The function $u^0 \in R_h(T, m_1, m_2)$ on which μ^0 is attained is unique (with the accuracy of up to the displacements with respect to time) and defined by the equations

$$u^0(t) = m_2, \quad 0 \leqslant t < \frac{h + m_1}{m_2 + m_1}T \tag{3.5}$$
$$u^0(t) = -m_1, \quad \frac{h + m_1}{m_2 + m_1}T \leqslant t < T$$

If $M_1 < 0$, then the corresponding smallest general index of (3.1) is

$$\mu^0 = \min_k \frac{k}{T}\ln\left|A_k + \sqrt{A_k^2 - 1}\right| \tag{3.6}$$
$$A_k = \operatorname{ch}\left(\frac{\sigma_1 T}{k}\right)\cos\left(\frac{\sigma_2 T}{k}\right) + \gamma\operatorname{sh}\left(\frac{\sigma_1 T}{k}\right)\sin\left(\frac{\sigma_2 T}{k}\right), \quad k = 1, 2, \ldots$$

The corresponding function u° is defined by the formulas (3.5) in which T has been replaced by T/k (k is a natural number ensuring the minimum in (3.6)).

An undefined parameter $h \in [-m_1, m_2]$ appears in (3.4)—(3.6). Let us choose h and the initial conditions for (3.2) so as to ensure the existence of a periodic solution of (3.2) for $u = u^\circ(t)$. To simplify the formulation of the problem we define the $OXYZ$ coordinate system so that $y(0) = 0$. Then the solution of (3.2) has the form

$$y^\cdot(t) = y^\cdot(0) + \int_0^t u^0(\tau)\,d\tau - ct \tag{3.7}$$

$$y(t) = y^\cdot(0)t + \int_0^t (t - \tau)u^\circ(\tau)\,d\tau - \frac{ct^2}{2} \tag{3.8}$$

For a T-periodic solution, $y^\cdot(T) = y^\cdot(0)$. Therefore, setting in (3.7) $t = T$, we obtain $h = c$. Further, since $y(T) = y(0) = 0$, and when $t = T$ in (3.8) yields

$$y^\cdot(0) = \frac{cT}{2} - \frac{1}{T}\int_0^T (T - \tau)u^\circ(\tau)\,d\tau = \frac{T(c - m_2)(c + m_1)}{2(m_1 + m_2)} \tag{3.9}$$

Next we find the extrema of the function $y(t)$ from (3.8) where $y^\cdot(0)$ is given by the formula (3.9) and $u^\circ(\tau)$ by the equations (3.5) with $h = c$. The extremal points represent the solutions of the equation $y^\cdot(t) = 0$. Carrying out the elementary manipulations we obtain

$$\min_t y(t) = -\frac{T^2}{8}\left(\frac{c + m_1}{m_2 + m_1}\right)^2(m_2 - c) \tag{3.10}$$
$$\max_t y(t) = \frac{T^2}{8}\left(\frac{m_2 - c}{m_2 + m_1}\right)^2(m_1 + c)$$

The condition (3.3) will hold if $\max y(t) \leqslant \varepsilon_0$ and $\min y(t) \geqslant -\varepsilon_0$. The above two inequalities can be replaced, according to (3.10), by a single inequality

$$T < 2\sqrt{2\varepsilon_0}\,\frac{m_1 + m_2}{B}, \quad B = \min\left\{(m_1 + c)\sqrt{m_2 - c},\ (m_2 - c)\sqrt{m_1 + c}\right\} \tag{3.11}$$

and in this manner we obtain the following result.

Theorem. If $M_1 > 0$, then the control $u^c(t)$ sought is given by the formula (3.5) with $h = c$. The minimum value of the general index is found from (3.4), and the period T must satisfy the inequality (3.11).

If $M_1 < 0$, then the control $u^\circ(t)$ sought is found from (3.5) with $h = c$, where T is replaced by T/k (k is a natural number ensuring the minimum in (3.6)). The minimum value of the general index is given by (3.6), and the period T/k must also satisfy (3.11).

Let us obtain an estimate for the fundamental matrix $X(t)$ of the system (3.1). Let the system be stable at $u = u^\circ(t)$. In this case $\mu^\circ = 0$ ($\mu^\circ < 0$ is impossible since the system (3.1) cannot be asymptotically stable). This is clearly the case $M_1 < 0$ of the lemma. From (3.6) we obtain $|A_k + \sqrt{A_k^2 - 1}| = 1$, consequently $|A_k| < 1$ and both multipliers of the system (3.1) are equal to unity in modulo /7/. In this case the following inequality holds:

$$\| X(t) \| < C_0 (1 - A_k^2)^{-1/2}, \quad t \in [0, \infty) \tag{3.12}$$

where C_0 is a constant depending on the parameters a, b, m_1, m_2, h and T/k.

The estimate (3.12) is obtained as follows: we obtain the matrix $X(T/k)$ in explicit form (this can be done by virtue of the piecewise constancy of the function $u^\circ(t)$), then reduce the resulting second order matrix to diagonal form and raise it to an arbitrary power n. Analogous estimates can easily be obtained for the case $\mu^\circ > 0$. The above estimates are necessary for checking the correctness of the linearization of the initial nonlinear equations.

4. Solution of the problems of Sect.2. We use the theorem of Sect.3 to solve the problems of Sect.2.

Pendulum with movable point of suspension. Carrying out in (2.2) the substitution $\varphi = x \exp(-{}^1/_2 k_1 t)$, we obtain the following equation for x:

$$x^{\cdot\cdot} = \left[\frac{F(t)}{l(m+M)} + \frac{k_1^2}{4}\right] x \tag{4.1}$$

Let us set $u(t) = F(t)/(m+M)$. Then the equations (4.1) and (2.1) become equivalent to the equations (3.1) and (3.2), provided that the parameters have the following values:

$$m_i = \frac{F_i}{m+M} \ (i = 1, 2), \quad c = g, \quad a = \frac{k_1^2}{4}, \quad b = \frac{1}{l} \tag{4.2}$$

Substituting now (4.2) into (3.4)—(3.6) and (3.11), we obtain the solution of the problem. The necessary and sufficient condition for the pendulum to be asymptotically stable under the constraints shown is that the inequality

$$\mu^\circ < {}^1/_2 k_1 \tag{4.3}$$

holds where μ° is given by the formula (3.4) or (3.6) with the parameters given by (4.2). Moreover, since the linearized equation (2.2) becomes asymptotically stable when (4.3) holds, it follows, in accordance with the first Liapunov method /7/, that the solution $\varphi = \varphi^\cdot = 0$ of the corresponding nonlinear equation (1.2) is also asymptotically stable.

Pendulum with movable rods. The construction in Fig.2 is described by the linearized equations (2.4) and (2.5). Let us make in (2.4) the substitution

$$\varphi = x \exp\left(-\frac{k}{2LR} t\right), \quad R = \frac{ML}{3} + 2mL$$

This yields the following equation for x:

$$x^{\cdot\cdot} = \left[-\frac{3}{2}\frac{M^\circ(t)}{lR} + \frac{M+m}{2R} g + \frac{k^2}{4L^2R^2}\right] x \tag{4.4}$$

We denote $u(t) = -{}^3/_4 M^\circ(t)/(ml^2)$, whereupon the equations (4.4) and (2.5) become equivalent to the equations (3.1) and (3.2) when the values of the parameters are

$$m_1 = \frac{3M_2^\circ}{4ml^2}, \quad m_2 = \frac{3M_1^\circ}{4ml^2}, \quad c = \frac{3g}{4l} \quad a = \frac{M+m}{2R} g + \frac{k^2}{4L^2R^2}, \quad b = \frac{2lm}{R} \tag{4.5}$$

Substituting the parameters given in (4.5) into (3.4)—(3.6) and (3.11), we obtain the solution of the problem. Moreover, the necessary and sufficient condition for the rod OD to be asymptotically stable is, that the inequality

$$\mu^\circ < k/(2LR) \tag{4.6}$$

holds where μ° is given by the formula (3.4) or (3.6), with the parameters taken from (4.5).

The construction in Fig.3 is described by the linearized equations (2.6) and (2.5). Let us make the following substitution in (2.6):

$$\varphi = x \exp\left[-\frac{k}{2LR}t - \frac{2ml}{R}\Delta\psi(t)\right]$$

and insert, into the resulting equation for x, the expression for $\Delta\psi^{\cdot\cdot}$ from (2.5)

$$x^{\cdot\cdot} = \left[-\frac{3M^0(t)}{2lR} + \frac{M+m}{2R}g + \left(\frac{k}{2LR} + \frac{2ml}{R}\Delta\psi^{\cdot}\right)^2\right]x \quad (4.7)$$

According to the formulas (3.7) and (3.9), the quantity $\Delta\psi^{\cdot}$ is small when the periods T are small (we assume that $\Delta\psi \equiv y(t)$). Consequently, when the rods DA and DA' oscillate at high frequencies, we can assume $\Delta\psi^{\cdot} = 0$. in (4.7) and this reduces it to (4.4). The first Liapunov method /7/ implies then when (4.6) holds, the solution $\varphi = \varphi^{\cdot} = 0$ of the initial nonlinear equation (1.3) or (1.5) is asymptotically stable. Inequalities of the type (3.12) should be used to check the correctness of the linearization.

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